AN APPLICATION OF A LOCAL VERSION OF CHANG'S THEOREM

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ABSTRACT. Suppose that G is a compact Abelian group. If $A \subset G$ then how small can $\|\chi_A\|_{A(G)}$ be? In general there is no non-trivial lower bound.

In [5] Green and Konyagin showed that if $\widehat{G}$ has sparse small subgroup structure and A has density α with $\alpha(1-\alpha) \gg 1$ then $\|\chi_A\|_{A(G)}$ does admit a non-trivial lower bound. To complement this [11] addressed the case where $\widehat{G}$ has rich small subgroup structure and further claimed a result for general compact Abelian groups. In this note we prove this claim by fusing the techniques of [5] and [11] in a straightforward fashion.

1. NOTATION AND INTRODUCTION

We use the Fourier transform on compact Abelian groups, the basics of which may be found in Chapter 1 of Rudin [9]; we take a moment to standardize our notation.

Suppose that G is a compact Abelian group. Write $\widehat{G}$ for the dual group, that is the discrete Abelian group of continuous homomorphisms $\gamma : G \to S^1$, where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. G may be endowed with Haar measure μ_G normalised so that $\mu_G(G) = 1$ and as a consequence we may define the Fourier transform $\widehat{\cdot} : L^1(G) \to \ell^\infty(\widehat{G})$ which takes $f \in L^1(G)$ to

$$\widehat{f} : \widehat{G} \to \mathbb{C}; \gamma \mapsto \int_{x \in G} f(x)\overline{\gamma(x)}d\mu_G(x).$$

We write

$$A(G) := \{f \in L^1(G) : \|\widehat{f}\|_1 < \infty\},$$

and define a norm on $A(G)$ by $\|f\|_{A(G)} := \|\widehat{f}\|_1$.

The following result is well known.

Proposition 1.1. *Suppose that G is a compact Abelian group. Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ we have[1] $\{\alpha|V|\}(1 - \{\alpha|V|\}) > 0$. Then $\chi_A \notin A(G)$.*

In [11] the following quantitative version of the above result was claimed.

Theorem 1.2. *Suppose that G is a compact Abelian group. Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ with $|V| \leq M$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg 1$. Then*

$$\|\chi_A\|_{A(G)} \gg \log\log\log M.$$

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[1]Here $\{x\}$ denoted the fractional part of x.

The objective of these notes is to prove Theorem 1.2; for completeness and to illuminate the arguments we begin with a proof of Proposition 1.1. In fact Proposition 1.1 follows is a straightforward corollary of the celebrated idempotent theorem of Cohen [4] and, indeed, the instance of the idempotent theorem which we use was in fact proved even earlier by Rudin in [8].

2. Proof of Proposition 1.1

We shall prove the following stronger result which characterizes those sets $A \subset G$ for which $\chi_A \in A(G)$.

Proposition 2.1. *Suppose that G is a compact Abelian group and $A \subset G$ has $\chi_A \in A(G)$. Then there is a finite $V \leq \widehat{G}$ such that $\chi_A = \chi_A * \mu_{V^\perp}$ almost everywhere.*

This result is a special case of the idempotent theorem for discrete groups first proved by Rudin in [8]. The general case (locally compact abelian groups) was proved by Cohen in [4].

Proof of Proposition 1.1. If $\chi_A \in A(G)$ then by Proposition 2.1 there is some finite $V \leq \widehat{G}$ such that $\chi_A = \chi_A * \mu_{V^\perp}$ almost everywhere. For each $W \in G/V^\perp$ pick $x_W \in W$ such that $\chi_A * \mu_{V^\perp}(x) = \chi_A(x_W)$ for almost all $x \in W$. Integrating tells us that

$$\alpha = \int \chi_A d\mu_G = \int \chi_A * \mu_{V^\perp} d\mu_G = |V|^{-1} \sum_{W \in G/V^\perp} \chi_A(x_W) = n|V|^{-1}$$

for some integer n. It follows that $\{\alpha|V|\}(1 - \{\alpha|V|\}) = 0$ which contradicts the hypothesis on α; hence $\chi_A \notin A(G)$. □

We use Bohr neighborhoods to prove Proposition 2.1; these will also be useful in the proof of Theorem 1.2.

2.2. **Bohr neighborhoods.** The following defines a natural valuation on S^1

$$\|z\| := \frac{1}{2\pi} \inf_{n \in \mathbb{Z}} |2\pi n + \arg z|,$$

which can be used to measure how far $\gamma(x)$ is from 1. This leads to a definition of approximate annihilators for a finite collection of characters Γ, namely

$$B(\Gamma, \delta) := \{x \in G : \|\gamma(x)\| \leq \delta \text{ for all } \gamma \in \Gamma\};$$

such sets are called *Bohr sets*, and their translates *Bohr neighborhoods*.

The following simple application of the pigeon-hole principle (the details for which may be found, for example, in [5]) shows that Bohr sets have positive measure.

Lemma 2.3. *Suppose G is a compact Abelian group and $B(\Gamma, \delta)$ is a Bohr set. Then $\mu_G(B(\Gamma, \delta)) \geq \delta^d$ where $d := |\Gamma|$.*

Bohr sets are important because the translates of a Bohr sets $B(\Gamma, \delta)$ are approximate joint level sets for the characters in Γ and hence a function that has the bulk of its Fourier transform supported on a finite set Γ is approximately constant on all translates of $B(\Gamma, \delta)$. Concretely we have the following lemma.

Lemma 2.4. *Suppose that G is a compact Abelian group and $f \in A(G)$. Suppose that $\eta \in (0,1]$. Then there is a finite set of characters Γ and a function g such that*

$$\sup_{x-y \in B(\Gamma, \eta/3\|f\|_{A(G)})} |g(x) - g(y)| \leq \eta \text{ and } f = g \text{ a.e.}$$

Proof. $f \in A(G)$ so there is a finite set of characters Γ such that

$$(2.1) \qquad \sum_{\gamma \notin \Gamma} |\widehat{f}(\gamma)| \leq \eta/3.$$

Write

$$g(x) := \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma)\gamma(x) \text{ and } g_\Gamma(x) := \sum_{\gamma \in \Gamma} \widehat{f}(\gamma)\gamma(x).$$

Now

$$|g_\Gamma(x) - g_\Gamma(y)| \leq \|f\|_{A(G)} \sup_{\gamma \in \Gamma} |\gamma(x) - \gamma(y)| \leq \|f\|_{A(G)} \sup_{\gamma \in \Gamma} \|\gamma(x-y)\| \leq \eta/3$$

if $x \in y + B(\Gamma, \eta/3\|f\|_{A(G)})$. From our choice of Γ in (2.1) we have that $|g(x) - g_\Gamma(x)| \leq \eta/3$ for all $x \in G$ so that

$$|g(x) - g(y)| \leq |g(x) - g_\Gamma(x)| + |g_\Gamma(x) - g_\Gamma(y)| + |g_\Gamma(y) - g(y)| \leq \eta$$

for all $x \in y + B(\Gamma, \eta/3\|f\|_{A(G)})$. By the inversion formula $f = g$ almost everywhere, from which the lemma follows. $\qquad\square$

If f in the above lemma is the characteristic function of a set then it takes only the values 0 or 1. We can use the following version of the intermediate value theorem (implicit in [5] and [13]) to show that such an f must be constant on cosets of $\langle B(\Gamma, \eta/3\|f\|_{A(G)}) \rangle$.

Lemma 2.5. (Discrete intermediate value theorem) *Suppose that G is a compact Abelian group and that $B(\Gamma, \delta)$ is a Bohr set on G. Suppose that $g : G \to \mathbb{R}$ has*

$$(2.2) \qquad \sup_{x-y \in B(\Gamma, \delta)} |g(x) - g(y)| \leq \eta.$$

Suppose that $x_0, x_1 \in G$ have $x_0 - x_1 \in \langle B(\Gamma, \delta) \rangle$. Then for any $c \in [g(x_0), g(x_1)]$, there is an $x_2 \in G$ such that

$$|g(x_2) - c| \leq \frac{\eta}{2}.$$

Proof. We write H for the group, $\langle B(\Gamma, \delta) \rangle$, generated by $B(\Gamma, \delta)$ and define

$$S^- := \{x \in x_0 + H : g(x) < c - \frac{\eta}{2}\}$$

and

$$S^+ := \{x \in x_0 + H : g(x) > c + \frac{\eta}{2}\}.$$

If the conclusion of the lemma is false then $\mathcal{S} := \{S^-, S^+\}$ is a partition of $x_0 + H$.

By the continuity hypothesis (2.2) we have that if $x \in S^-$ and $y \in B(\Gamma, \delta)$ then

$$|g(x+y) - g(x)| \leq \eta \Rightarrow g(x+y) < c + \frac{\eta}{2}.$$

It follows that $x + y \notin S^+$ and since $\mathcal{S}$ is a partition of $x_0 + H$ we conclude that $x + y \in S^-$. We have shown that $S^- = S^- + H$.

Now $g(x_0) \leq c \leq g(x_1)$ and $\mathcal{S}$ is a partition of $x_0 + H$, whence $x_0 \in S^-$ and $x_1 \in S^+$. However $S^- = S^- + H$, whence $S^- = x_0 + H$ and so $x_1 \in S^-$. This contradicts the fact that S^- and S^+ are disjoint and so proves the lemma. $\qquad\square$

Proof of Proposition 2.1. Apply Lemma 2.4 to χ_A with $\eta = 1/4$. The lemma provides a function g with

$$(2.3) \qquad \sup_{x-y \in B(\Gamma, 1/12\|\chi_A\|_{A(G)})} |g(x) - g(y)| \leq 1/4 \text{ and } g = \chi_A \text{ a.e.}$$

Write $H := \langle B(\Gamma, 1/12\|\chi_A\|_{A(G)}) \rangle$.

Claim 2.5.1. *χ_A is constant on cosets of H (up to a null set).*

Proof. Suppose that W is a coset of H and there is a subset of W of positive measure on which χ_A is 0 and a subset of W of positive measure on which χ_A is 1. Since $g = \chi_A$ a.e. it follows that there are $x_0, x_1 \in W$ (so that $x_0 - x_1 \in H$) with $g(x_0) = 0$ and $g(x_1) = 1$.

By Lemma 2.5 there is some $x_2 \in G$ such that $|g(x_2) - 1/2| \leq 1/8$ and the fact that g satisfies (2.3) then ensures that $|g(x) - 1/2| \leq 3/8$ for all $x \in x_2 + B(\Gamma, 1/12\|\chi_A\|_{A(G)})$. By Lemma 2.3 we know that $x_2 + B(\Gamma, 1/12\|\chi_A\|_{A(G)})$ as positive measure so that there is some $x \in x_2 + B(\Gamma, 1/12\|\chi_A\|_{A(G)})$ such that $g(x) = \chi_A(x) \in \{0, 1\}$. This contradicts the fact that $1/8 \leq g(x) \leq 7/8$. The claim follows. $\qquad \square$

Again by Lemma 2.3 we have that $\mu_G(H) \geq \mu_G(B(\Gamma, 1/12\|\chi_A\|_{A(G)})) > 0$ so that $V := H^\perp$ is finite. Since $V^\perp = H$ the proposition is proved. $\qquad \square$

The remainder of these notes establishes Theorem 1.2. §3 recalls some basic tools of local Fourier analysis. Roughly §4 provides an effective quantitative version of Lemma 2.4 and §5 establishes the necessary physical space estimates to drive the result of §4. §6 combines the results of the previous two sections to prove the main result.

3. Local Fourier analysis on compact Abelian groups

Bourgain, in [1], observed that one can localize the Fourier transform to typical approximate level sets and retain approximate versions of a number of the standard results for the Fourier transform on compact Abelian groups. Since his original work various expositions and extensions have appeared most notably in the various papers of Green and Tao. We require a local version of Chang's theorem as developed in [12]; we follow the preparatory discussion in there fairly closely.

3.1. **Approximate annihilators: typical Bohr sets and some of their properties.** We defined Bohr sets in §2.2 and in view of Lemma 2.3 we write $\beta_{\Gamma,\delta}$, or simply β or β_δ if the parameters are implicit, for the measure induced on $B(\Gamma, \delta)$ by μ_G, normalised so that $\|\beta_{\Gamma,\delta}\| = 1$. Such measures are sometimes referred to as *normalised Bohr cutoffs*. We write β' for $\beta_{\Gamma',\delta'}$, or $\beta_{\Gamma,\delta'}$ if no Γ' has been defined. Having defined these measures we norm the L^p-spaces $L^p(B(\Gamma, \delta))$ in the obvious way *viz.*

$$\|f\|_{L^p(B(\Gamma,\delta))} := \left(\int |f|^p d\beta_{\Gamma,\delta} \right)^{\frac{1}{p}}.$$

As we noted Bohr sets can be thought of as approximate annihilators, however genuine annihilators are also subgroups of G, a property which, at least in an approximate form, we would like to recover. Suppose that $\eta \in (0, 1]$. Then $B(\Gamma, \delta) + B(\Gamma, \eta\delta) \subset B(\Gamma, (1+\eta)\delta)$. If $B(\Gamma, (1+\eta)\delta)$ is not much bigger than $B(\Gamma, \delta)$ then we have a sort of approximate additive closure in the sense that $B(\Gamma, \delta) + B(\Gamma, \eta\delta) \approx$

$B(\Gamma, (1+\eta)\delta)$. Not all Bohr sets have this property however Bourgain showed that typically they do. For our purposes we have the following proposition.

Proposition 3.2. *Suppose that G is a compact Abelian group, Γ a set of d characters on G and $\delta \in (0,1]$. There is an absolute constant $c_{\mathcal{R}} > 0$ and a $\delta' \in [\delta/2, \delta)$ such that*

$$(3.1) \qquad \frac{\mu_G(B(\Gamma, (1+\kappa)\delta'))}{\mu_G(B(\Gamma, \delta'))} = 1 + O(|\kappa|d)$$

whenever $|\kappa|d \le c_{\mathcal{R}}$.

This result is not as easy as the rest of the section, it uses a covering argument; a nice proof can be found in [7]. We say that δ' is *regular for* Γ or that $B(\Gamma, \delta')$ is *regular* if

$$\frac{\mu_G(B(\Gamma, (1+\kappa)\delta'))}{\mu_G(B(\Gamma, \delta'))} = 1 + O(|\kappa|d) \text{ whenever } |\kappa|d \le c_{\mathcal{R}}.$$

It is regular Bohr sets to which we localize the Fourier transform and we begin by observing that normalised regular Bohr cutoffs are approximately translation invariant and so function as normalised approximate Haar measures.

Lemma 3.3. (Normalized approximate Haar measure) *Suppose that G is a compact Abelian group and $B(\Gamma, \delta)$ is a regular Bohr set. If $y \in B(\Gamma, \delta')$ then $\|(y + \beta_\delta) - \beta_\delta\| \ll d\delta'\delta^{-1}$ where we recall that $y + \beta_\delta$ denotes the measure β_δ composed with translation by y.*

The proof follows immediately from the definition of regularity. In applications the following corollary will be useful but it should be ignored until it is used.

Corollary 3.4. *Suppose that G is a compact Abelian group and $B(\Gamma, \delta)$ is a regular Bohr set. If $f \in L^\infty(G)$ then*

$$\|f * \beta - f * \beta(x)\|_{L^\infty(x + B(\Gamma, \delta'))} \ll \|f\|_{L^\infty(G)} d\delta'\delta^{-1}.$$

With an approximate Haar measure we are in a position to define the local Fourier transform: Suppose that $x' + B(\Gamma, \delta)$ is a regular Bohr neighborhood. Then we define the Fourier transform local to $x' + B(\Gamma, \delta)$ by

$$L^1(x' + B(\Gamma, \delta)) \to \ell^\infty(\widehat{G}); f \mapsto \widehat{fd(x' + \beta_{\Gamma,\delta})}.$$

3.5. **The structure of sets of characters supporting large values of the local Fourier transform.** Having defined the local transform and recorded the key tools it remains for us to recall the result from [12] to which the title of these notes refers.

Proposition 3.6. (Chang's theorem local to Bohr sets, [12], Proposition 5.2) *Suppose that G is a compact Abelian group and $B(\Gamma, \delta)$ is a regular Bohr set. Suppose that $f \in L^2(B(\Gamma, \delta))$ and $\epsilon, \eta \in (0,1]$. Then there is a set of characters Λ and a $\delta' \in (0,1]$ with*

$$|\Lambda| \ll \epsilon^{-2} \log \|f\|_{L^1(B(\Gamma,\delta))}^{-2} \|f\|_{L^2(B(\Gamma,\delta))}^2$$

and

$$\delta' \gg \delta\eta\epsilon^2/d^2 \log \|f\|_{L^1(B(\Gamma,\delta))}^{-2} \|f\|_{L^2(B(\Gamma,\delta))}^2$$

and furthermore,

$$\{\gamma \in \widehat{G} : |\widehat{fd\beta}(\gamma)| \ge \epsilon \|f\|_{L^1(B(\Gamma,\delta))}\}$$

is contained in

$$\{\gamma \in \widehat{G} : |1 - \gamma(x)| \leq \eta \ for \ all \ x \in B(\Gamma \cup \Lambda, \delta')\}.$$

4. AN ITERATION ARGUMENT IN FOURIER SPACE

The main result of this section takes physical space information about a set $A \subset G$ and converts it into Fourier information. The lemma is based on Lemma 4.8 in [11] with two main modifications:

- We have to assume the comparability of the local L^2-norm squared and local L^1-norm; ensuring this hypothesis is the principal extra complication of §5.
- We are less careful in our analysis because the physical space estimates available to us in the general setting are sufficiently weak as to render any more care irrelevant.

Lemma 4.1. (Iteration lemma) *Suppose that G is a compact Abelian group, $B(\Gamma, \delta)$ is a Bohr set and $B(\Gamma, \delta')$ is a regular Bohr set. Suppose that $A \subset G$ has $\chi_A \in A(G)$ and write $f := \chi_A - \chi_A * \beta$. Suppose, additionally, that*

$$\|f\|^2_{L^2(B(\Gamma, \delta'))} \asymp \|f\|_{L^1(B(\Gamma, \delta'))} \ and \ \|f\|^2_{L^2(B(\Gamma, \delta'))} > 0.$$

Suppose that $\epsilon \in (0, 1]$ is a parameter. Then either $\|\chi_A\|_{A(G)} \gg \epsilon^{-1}$ or there is a set of characters Λ and a regular Bohr set $B(\Gamma \cup \Lambda, \delta'')$ such that

$$|\Lambda| \ll \epsilon^{-2} \log \|f\|^{-1}_{L^2(B(\Gamma, \delta'))} \ and \ \delta'' \gg \delta' \epsilon^3 / d^2 \log \|f\|^{-1}_{L^2(B(\Gamma, \delta'))},$$

where, as usual, $d := |\Gamma|$, and

$$\sum_{\gamma \in \mathcal{N} \setminus \mathcal{O}} |\widehat{\chi_A}(\gamma)| \gg 1,$$

where $\mathcal{O} := \{\gamma : |1 - \gamma(x)| \leq \epsilon \ for \ all \ x \in B(\Gamma, \delta)\}$ and $\mathcal{N} := \{\gamma : |1 - \gamma(x)| \leq \epsilon \ for \ all \ x \in B(\Gamma \cup \Lambda, \delta'')\}$.

Proof. By Plancherel's theorem we have

$$(4.1) \qquad \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma) \overline{\widehat{fd\beta'}(\gamma)} = \|f\|^2_{L^2(B(\Gamma, \delta'))}.$$

Write

$$\mathcal{L} := \{\gamma : |\widehat{fd\beta'}(\gamma)| \geq \epsilon \|f\|_{L^1(B(\Gamma, \delta'))}\},$$

and suppose that

$$(4.2) \qquad \sum_{\gamma \notin \mathcal{L}} \widehat{f}(\gamma) \overline{\widehat{fd\beta'}(\gamma)} \geq \|f\|^2_{L^2(B(\Gamma, \delta'))}/2.$$

Note that

$$\|f\|_{A(G)} = \|\chi_A - \chi_A * \beta\|_{A(G)} \leq \|\chi_A\|_{A(G)} + \|\chi_A * \beta\|_{A(G)} \leq 2\|\chi_A\|_{A(G)},$$

whence

$$\begin{aligned}
\sum_{\gamma \notin \mathcal{L}} |\widehat{f}(\gamma)||\widehat{fd\beta'}(\gamma)| \quad &\leq \quad \epsilon \|f\|_{L^1(B(\Gamma, \delta'))} \|f\|_{A(G)} \\
&\leq \quad 2\epsilon \|\chi_A\|_{A(G)} \|f\|_{L^1(B(\Gamma, \delta'))} \\
&\ll \quad \epsilon \|\chi_A\|_{A(G)} \|f\|^2_{L^2(B(\Gamma, \delta'))}.
\end{aligned}$$

If (4.2) holds then the left hand side of this is at least $\|f\|_{L^2(B(\Gamma,\delta'))}^2/2$ and so (dividing by $\|f\|_{L^2(B(\Gamma,\delta'))}^2$) we conclude that $\|\chi_A\|_{A(G)} \gg \epsilon^{-1}$.

Thus we may suppose that (4.2) is not true and therefore, by (4.1), that

$$\sum_{\gamma \in \mathcal{L}} \widehat{f}(\gamma)\overline{\widehat{fd\beta'}(\gamma)} \geq \|f\|_{L^2(B(\Gamma,\delta'))}^2/2.$$

By Proposition 3.6 there is a set of characters Λ and a δ'' (regular for $\Gamma \cup \Lambda$ by Proposition 3.2), with

$$|\Lambda| \ll \epsilon^{-2}\log\|f\|_{L^1(B(\Gamma,\delta'))}^{-2}\|f\|_{L^2(B(\Gamma,\delta'))}^2 \ll \epsilon^{-2}\log\|f\|_{L^2(B(\Gamma,\delta'))}^{-1}$$

and

$$\delta'' \gg \delta'\epsilon^3/d^2\log\|f\|_{L^1(B(\Gamma,\delta'))}^{-2}\|f\|_{L^2(B(\Gamma,\delta'))}^2 \gg \delta'\epsilon^3/d^2\log\|f\|_{L^1(B(\Gamma,\delta'))}^{-1},$$

such that

$$\mathcal{L} \subset \{\gamma : |1 - \gamma(x)| \leq \epsilon \text{ for all } x \in B(\Gamma \cup \Lambda, \delta'')\} = \mathcal{N}.$$

Since $\mathcal{L} \subset \mathcal{N}$ we have

$$\sum_{\gamma \in \mathcal{N}} |\widehat{f}(\gamma)\widehat{fd\beta'}(\gamma)| \geq \|f\|_{L^2(B(\Gamma,\delta'))}^2/2.$$

Now

$$|\widehat{fd\beta'}(\gamma)| \leq \|f\|_{L^1(B(\Gamma,\delta'))} \ll \|f\|_{L^2(B(\Gamma,\delta'))}^2,$$

hence

$$(4.3) \qquad\qquad \sum_{\gamma \in \mathcal{N}} |\widehat{f}(\gamma)| \gg 1.$$

Finally suppose that

$$(4.4) \qquad\qquad \sum_{\gamma \in \mathcal{O}} |\widehat{f}(\gamma)| \geq \frac{1}{2}\sum_{\gamma \in \mathcal{N}} |\widehat{f}(\gamma)|.$$

By the definition of $\mathcal{O}$ we have

$$\sum_{\gamma \in \mathcal{O}} |\widehat{f}(\gamma)| = \sum_{\gamma \in \mathcal{O}} |\widehat{\chi_A}(\gamma)|.|1 - \widehat{\beta}(\gamma)| \leq \|\chi_A\|_{A(G)} \sup_{\gamma \in \mathcal{O}} |1 - \widehat{\beta}(\gamma)| \leq \epsilon\|\chi_A\|_{A(G)}.$$

It follows that if (4.4) holds then, in view of (4.3), $\|\chi_A\|_{A(G)} \gg \epsilon^{-1}$. Thus we may assume it does not and hence that

$$\sum_{\gamma \in \mathcal{N} \setminus \mathcal{O}} |\widehat{f}(\gamma)| \gg 1.$$

Noting that $|\widehat{f}(\gamma)| \leq 2|\widehat{\chi_A}(\gamma)|$ completes the proof. $\qquad\square$

5. Physical space estimates

The objective of this section is to prove the following result.

Proposition 5.1. *Suppose that G is a finite Abelian group and $B(\Gamma, \delta)$ is a regular Bohr set in G. Suppose that $A \subset G$ has density α and for all finite $V \leq \widehat{G}$ with $|V| \leq M$ we have $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg 1$. Then either*

$$\log M \ll d(\log\delta^{-1} + d\log d)$$

or there is an $x'' \in G$ and reals δ' and δ'', both regular for Γ, with $\delta' \leq \delta$,

$$\log\delta\delta'^{-1} \ll d\log d \text{ and } \log\delta''^{-1} \ll d(\log\delta^{-1} + d\log d)$$

such that

$$(5.1) \qquad \|\chi_A - \chi_A * \beta'\|^2_{L^2(x''+B(\Gamma,\delta''))} \asymp \|\chi_A - \chi_A * \beta'\|_{L^1(x''+B(\Gamma,\delta''))}$$

and

$$(5.2) \qquad \log \|\chi_A - \chi_A * \beta'\|^{-2}_{L^2(x''+B(\Gamma,\delta''))} \ll d(\log \delta^{-1} + d \log d).$$

Of the two parts (5.1) and (5.2) the second is the easiest to derive and comes essentially from a straightforward generalization of the physical space estimates of [11] combined with discrete intermediate value theorem (Lemma 2.5). To ensure (5.1) requires more work and is the principal extra ingredient of these notes.

We begin with a version of Lemma 5.1, [11], appropriate to our more general setting. In fact the proof which follows is slightly simpler than that in [11] and would have been sufficient for application there as well; the weakness of the present approach only impacts on the implied constants.

Lemma 5.2. *Suppose that G is a finite Abelian group. Suppose that $f \in L^1(G)$ maps G into $[0,1]$ and that $V \leq \widehat{G}$, finite, has $\{\|f\|_1|V|\}(1-\{\|f\|_1|V|\}) \gg 1$. Then there is a coset $x' + V^\perp$ with*

$$f * \mu_{V^\perp}(x') \gg \mu_G(V^\perp) \text{ and } (1-f) * \mu_{V^\perp}(x') \gg \mu_G(V^\perp).$$

Proof. $f * \mu_{V^\perp}$ is constant on cosets of $V^\perp$ so we define

$$g(x) := \begin{cases} 1 & \text{if } f * \mu_{V^\perp}(x) \geq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Since g is integral on cosets of $V^\perp$ there is some integer n such that

$$\int g d\mu_G = n\mu_G(V^\perp).$$

However

$$\begin{aligned}
|n\mu_G(V^\perp) - \|f\|_1| &= |\int g d\mu_G - \int f d\mu_G| \\
&= |\int g d\mu_G - \int f * \mu_{V^\perp} d\mu_G| \\
&\leq \int |g - f * \mu_{V^\perp}| d\mu_G \\
&\leq \sup_{x \in G} \min\{f * \mu_{V^\perp}(x), 1 - f * \mu_{V^\perp}(x)\} \\
&= \sup_{x \in G} \min\{f * \mu_{V^\perp}(x), (1-f) * \mu_{V^\perp}(x)\}.
\end{aligned}$$

Now

$$\begin{aligned}
|n\mu_G(V^\perp) - \|f\|_1| &= \mu_G(V^\perp)|n - |V|\|f\|_1| \\
&\geq \mu_G(V^\perp)\{\|f\|_1|V|\}(1-\{\|f\|_1|V|\}) \\
&\gg \mu_G(V^\perp),
\end{aligned}$$

and the conclusion of the lemma follows. $\qquad\qquad\square$

We require one more preliminary lemma.

Lemma 5.3. *Suppose that G is a finite Abelian group and $B(\Gamma, \delta)$ is a Bohr set in G. Then there are reals δ' and δ'' both regular for Γ with $\delta' \leq \delta$,*

$$\log \delta \delta'^{-1} \ll d \log d \text{ and } \delta'' \gg \delta'/d$$

such that

$$B(\Gamma, \delta')^{\perp} = B(\Gamma, \delta'')^{\perp}$$

and

$$\|f * \beta' - f * \beta'(x)\|_{L^{\infty}(x + B(\Gamma, \delta''))} \leq \|f\|_{\infty}/4$$

for all $x \in G$ and $f \in L^{\infty}(G)$.

Proof. We define a sequence $(\delta_i)_i$ iteratively and write

$$\beta_i := \beta_{\Gamma, \delta_i} \text{ and } H_i := \langle B(\Gamma, \delta_i) \rangle.$$

To begin with we apply Proposition 3.2 to get some δ_0 regular for Γ with $\delta \geq \delta_0 \gg \delta$. Now, if we have constructed δ_i for some $i \geq 0$, we apply Corollary 3.4 (and Proposition 3.2) to get a δ_{i+1} regular for Γ with $\delta_i \geq \delta_{i+1} \gg \delta_i/d$ and

$$\|f * \beta_i - f * \beta_i(x)\|_{L^{\infty}(x + B(\Gamma, \delta_{i+1}))} \leq \|f\|_{\infty}/4$$

for all $x \in G$ and $f \in L^{\infty}(G)$. We are done if we can show that there is some $i \leq d$ such that $H_i = H_{i+1}$. This follows by the pigeon-hole principle from the following claim.

Claim 5.3.1. *Suppose that $\kappa_0 \in (0, 1]$. Then there is a sequence of elements $x_1, ..., x_d \in G$ such that for each $\kappa \in (\kappa_0, 1]$ there is some $0 \leq i \leq d$ such that*

$$\langle B(\Gamma, \kappa) \rangle = \langle x_1, ..., x_i \rangle + \bigcap_{\gamma \in \Gamma} \ker \gamma.$$

Proof. The proof of the claim is based on ideas from the geometry of numbers introduced to the area by Ruzsa in [10]; [6] contains a neat exposition of the idea. By quotienting we may assume that $\bigcap_{\gamma \in \Gamma} \ker \gamma = \{0_G\}$.

Let $\phi : G \to \mathbb{T}^d; x \mapsto (\gamma(x))_{\gamma \in \Gamma}$ and define the lattice $\mathcal{L} := \bigcup \phi(G) \leq \mathbb{R}^d$. Since $\bigcap_{\gamma \in \Gamma} \ker \gamma = \{0_G\}$ there is a natural homomorphism $\psi : \mathcal{L} \to G$ which takes $b \in \mathcal{L}$ to the unique $x \in G$ such that $\phi(x) = b + \mathbb{Z}^d$, with kernel $\mathbb{Z}^d$.

We write Q for the unit cube centered at the origin in $\mathbb{R}^d$ and note that $\psi(\kappa Q) = B(\Gamma, \kappa)$. We choose linearly independent vectors $b_1, .., b_d \in \mathcal{L}$ inductively so that

$$\|b_i\|_{\infty} \leq \inf\{\lambda : \lambda Q \cap \mathcal{L} \text{ contains } i \text{ linearly independent vectors}\}.$$

Let $x_i = \psi(b_i)$. Since ψ is linear we have $\langle B(\Gamma, \kappa) \rangle = \psi(\langle \kappa Q \rangle)$, but to each $\kappa \in (0, 1]$ there corresponds an $1 \leq i \leq d$ such that κQ contains at most i linearly independent vectors and κQ contains $b_1, ..., b_i$. Hence $\langle \kappa Q \rangle = \langle b_1, ..., b_i \rangle$. Linearity of ψ again gives the result. $\square$

$\square$

Proof of Proposition 5.1. Applying Lemma 5.3 we get reals $\delta' \leq \delta$ and δ''' both regular for Γ with

$$\log \delta \delta'^{-1} \ll d \log d \text{ and } \delta''' \gg \delta'/d$$

such that

$$B(\Gamma, \delta')^{\perp} = B(\Gamma, \delta''')^{\perp} =: V,$$

and

(5.3) $\|\chi_A * \beta' - \chi_A * \beta'(x)\|_{L^\infty(x+B(\Gamma,\delta'''))} \le 1/4$ for all $x \in G$.

Now

(5.4) $|V|^{-1} = \mu_G(V^\perp) \ge \mu_G(B(\Gamma,\delta')) \ge (\delta')^d;$

the last inequality by Lemma 2.3. If $M \le |V|$ then we are in the first case of the lemma. Otherwise by hypothesis $\{\alpha|V|\}(1 - \{\alpha|V|\}) \gg 1$. If we put $f = \chi_A * \beta'$ then $\|f\|_1 = \alpha$ and f maps G into $[0,1]$ whence, by Lemma 5.2, there is some $x''' \in G$ such that

(5.5) $\chi_A * \beta' * \mu_{V^\perp}(x''') \gg \mu_G(V^\perp)$ and $(1 - \chi_A * \beta') * \mu_{V^\perp}(x''') \gg \mu_G(V^\perp)$.

The argument now splits into three cases.

(1) *There are elements* $x_0, x_1 \in x''' + V^\perp$ *such that* $\chi_A * \beta'(x_0) \ge 1/2$ *and* $\chi_A * \beta'(x_1) \le 1/2$. Here $x_0 - x_1 \in V^\perp = B(\Gamma,\delta''')^{\perp\perp} = \langle B(\Gamma,\delta''')\rangle$, so by the discrete intermediate value theorem (Lemma 2.5) and (5.3) we conclude that there is some $x_2 \in x''' + V^\perp$ such that
$$\frac{3}{8} \le \chi_A * \beta'(x_2) \le \frac{5}{8}.$$

Further by (5.3) we conclude that
$$\frac{1}{8} \le \chi_A * \beta'(x) \le \frac{7}{8} \text{ for all } x \in x_2 + B(\Gamma,\delta''').$$

Since χ_A only takes values in $\{0,1\}$ it follows that $|\chi_A - \chi_A * \beta'| \asymp 1$ on $x_2 + B(\Gamma,\delta''')$. Thus
$$\|\chi_A - \chi_A * \beta'\|_{L^2(x_2+B(\Gamma,\delta'''))}^2 \asymp \mu_G(B(\Gamma,\delta''')),$$
and
$$\|\chi_A - \chi_A * \beta'\|_{L^1(x_2+B(\Gamma,\delta'''))} \asymp \mu_G(B(\Gamma,\delta''')).$$
The result follows on putting $\delta'' = \delta'''$; Lemma 2.3 then gives (5.2).

(2) $\chi_A * \beta'(x) \le 1/2$ *for all* $x \in x''' + V^\perp$. Suppose that $\delta'' \le \delta'$. Then $B(\Gamma,\delta'') \subset B(\Gamma,\delta')$ so we have $\langle B(\Gamma,\delta'')\rangle \subset \langle B(\Gamma,\delta')\rangle$, whence $\beta'' * \mu_{V^\perp} = \mu_{V^\perp} = \beta' * \mu_{V^\perp}$. Thus we define
$$\alpha' := \chi_A * \beta' * \mu_{V^\perp}(x''') = \chi_A * \beta'' * \mu_{V^\perp}(x'''),$$
which has
$$\alpha' \gg \mu_G(V^\perp) \ge (\delta')^d)^d > 0$$
by (5.5) and (5.4). By Corollary 3.4 we can pick a δ'' (regular for Γ by Proposition 3.2) with $\delta' \ge \delta'' \gg \delta'\alpha'/d$ such that

(5.6) $\|\chi_A - \chi_A * \beta'(x)\|_{L^\infty(x+B(\Gamma,\delta''))} \le \alpha'$ for all $x \in G$.

We write
$$L := \{x \in x''' + V^\perp : \chi_A * \beta''(x) \ge \alpha'/2\}$$
and note that
$$\int_{x \notin L} \chi_A * \beta''(x) d\mu_{V^\perp}(x''' - x) \le \sup_{x \notin L} \chi_A * \beta''(x) \le \alpha'/2,$$
so
$$\int_{x \in L} \chi_A * \beta''(x) d\mu_{V^\perp}(x''' - x) \ge \alpha'/2.$$

If $\chi_A * \beta''(x) \neq 0$ then $\chi_A * \beta'(x) \neq 0$, whence

$$\begin{aligned}
\alpha'/2 &\leq \int_{x \in L} \chi_A * \beta'(x) \frac{\chi_A * \beta''(x)}{\chi_A * \beta'(x)} d\mu_{V^\perp}(x''' - x) \\
&\leq \alpha' \sup_{x \in L} \frac{\chi_A * \beta''(x)}{\chi_A * \beta'(x)},
\end{aligned}$$

Dividing by α' (which we have previously observed is positive) we conclude that there is some $x'' \in L$ such that

$$\chi_A * \beta''(x'') \geq \chi_A * \beta'(x'')/4.$$

If $x \in A \cap (x'' + B(\Gamma, \delta''))$ then $|\chi_A(x) - \chi_A * \beta'(x)| \asymp 1$ since $\chi_A * \beta'(x) \leq 1/2$ by the hypothesis of this case. If $x \in A^c \cap (x'' + B(\Gamma, \delta''))$ then

$$|\chi_A(x) - \chi_A * \beta'(x)| \leq |\chi_A * \beta'(x)| \leq \chi_A * \beta'(x'') + O(\alpha') = O(\alpha'),$$

where the second inequality is a result of (5.6). It follows that

$$\|\chi_A - \chi_A * \beta'\|_{L^1(x'' + B(\Gamma, \delta''))} \gg \chi_A * \beta''(x''),$$

and

$$\begin{aligned}
\|\chi_A - \chi_A * \beta'\|_{L^1(x'' + B(\Gamma, \delta''))} &\leq O(\chi_A * \beta''(x'')) + O(\alpha') \\
&= O(\chi_A * \beta''(x''))
\end{aligned}$$

since $\chi_A * \beta''(x'') \gg \alpha'$ since $x'' \in L$. Similarly we have

$$\|\chi_A - \chi_A * \beta'\|^2_{L^2(x'' + B(\Gamma, \delta''))} \gg \chi_A * \beta''(x''),$$

and

$$\begin{aligned}
\|\chi_A - \chi_A * \beta'\|^2_{L^2(x'' + B(\Gamma, \delta''))} &\leq O(\chi_A * \beta''(x'')) + O(\alpha') \\
&= O(\chi_A * \beta''(x'')).
\end{aligned}$$

It follows that

$$\|\chi_A - \chi_A * \beta'\|_{L^1(x'' + B(\Gamma, \delta''))} \asymp \|\chi_A - \chi_A * \beta'\|^2_{L^2(x'' + B(\Gamma, \delta''))},$$

and

$$\|\chi_A - \chi_A * \beta'\|^2_{L^2(x'' + B(\Gamma, \delta''))} \gg \alpha'.$$

(3) $\chi_A * \beta'(x) \geq 1/2$ *for all* $x \in x''' + V^\perp$. This follows by replacing A in the previous case by A^c.

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6. PROOF OF THEOREM 1.2

Proof. In what follows it is convenient to let $C > 0$ denote an absolute constant which may vary from instance to instance.

Fix $\epsilon \in (0, 1]$ to be optimized later. We define three sequences $(\delta_k)_k$, $(\delta'_k)_k$ and $(\delta''_k)_k$ of reals, one sequence $(x_k)_k$ of elements of G, and one sequence $(\Gamma_k)_k$ of sets of characters inductively. We write

$$\beta_k := \beta_{\Gamma_k, \delta_k}, \beta'_k := \beta_{\Gamma_k, \delta'_k} \text{ and } \beta''_k := \beta_{\Gamma_k, \delta''_k},$$

as well as $d_k := |\Gamma_k|$ and

$$\mathcal{L}_k := \{\gamma : |1 - \gamma(x)| \leq \epsilon \text{ for all } x \in B(\Gamma_k, \delta_k)\}.$$

We shall ensure the following properties.

(1) $B(\Gamma_k, \delta_k)$, $B(\Gamma_k, \delta'_k)$ and $B(\Gamma_k, \delta''_k)$ are regular;

(2)
$$\|\chi_A - \chi_A * \beta_k\|^2_{L^2(x_k + B(\Gamma_k, \delta'_k))} \asymp \|\chi_A - \chi_A * \beta'_k\|_{L^1(x_k + B(\Gamma_k, \delta''_k))};$$

(3)
$$\|\chi_A - \chi_A * \beta'_k\|^2_{L^2(x_k + B(\Gamma_k, \delta''_k))} \gg \delta_k^{d_k}/(Cd_k)^{d_k^2};$$

(4)
$$\delta_k \geq \delta'_k \gg \delta_k/(Cd_k)^{d_k} \text{ and } \delta_k \geq \delta''_k \gg \delta_k^{d_k+1}/(Cd_k)^{d_k^2+d_k+1};$$

(5)
$$d_{k+1} \ll \epsilon^{-2} d_k (\log \delta_k^{-1} + d_k \log d_k);$$

(6)
$$\delta_k \geq \delta_{k+1} \gg \delta_k^{d_k+1} \epsilon^4/(Cd_k)^{d_k^2+d_k+6} \log \delta_k^{-1};$$

(7)
$$\sum_{\gamma \in \mathcal{L}_{k+1} \setminus \mathcal{L}_k} |\widehat{\chi_A}(\gamma)| \gg 1.$$

We initialize the iteration with $\Gamma_0 = \{0_{\widehat{G}}\}$. Pick $\delta_0 \gg 1$ regular for Γ_0 by Proposition 3.2. Apply Proposition 5.1 (assuming that $\delta_0^{d_0}(Cd_0)^{d_0^2} < M$) to get x_0 δ'_0 and δ''_0 satisfying properties (1),(2),(3) and (4). By translating A by $-x_0$, if necessary, we can apply Lemma 4.1 (assuming that we don't have $\|\chi_A\|_{A(G)} \gg \epsilon^{-1}$) to get Γ_1 and δ_1 such that properties (5), (6) and (7) are satisfied.

Given Γ_k and δ_k we can proceed as we just have (assuming that $\delta_k^{d_k}(Cd_k)^{d_k^2} < M$) to generate x_k, δ'_k, δ''_k, δ_{k+1} and Γ_{k+1}.

By property (7) (and the leftmost inequality in (6)) we have $\|\chi_A\|_{A(G)} \gg k$, so either $\|\chi_A\|_{A(G)} \gg \epsilon^{-1}$, or the iteration terminates with $k \ll \epsilon^{-1}$.

(5) and (6) imply
$$d_{k+1} \ll \epsilon^{-2} d_k \log \delta_k^{-1} \text{ and } \log \delta_{k+1}^{-1} \ll d_k \log \delta_k^{-1} + \log \epsilon^{-1},$$

whence
$$d_{k+1} \ll \epsilon^{-2} d_k \log \delta_k^{-1} \text{ and } \epsilon^{-2} \log \delta_{k+1}^{-1} \ll \epsilon^{-2} d_k \log \delta_k^{-1}.$$

It follows that $d_{k+1} \ll d_k^2$ and so $d_k \leq 2^{2^{Ck}}$ and $\delta_k \geq 2^{2^{2^{Ck}}}$. For the iteration to terminate we must have $M \leq \delta_k^{d_k}(Cd_k)^{d_k^2}$; for this to happen for some $k \ll \epsilon^{-1}$ we need $2^{2^{2^{C\epsilon^{-1}}}} \geq M$. The result follows. $\qquad\square$

7. Acknowledgements

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